Indian Statistical Institute, Bangalore

M. Math First Year

First Semester - Measure Theory

Midterm Exam — Date: 08th September 2025

Maximum marks: 30 — Duration: 2 hours

Answer any five, each question carries 6 marks.

1. (a) What is the σ-algebra generated by singletons. Justify your answer.

   (b) Let X be a set and $f\colon X \to [0,\infty]$ be a function. For any $E \subset X$, define $\mu(E) = 0$ if $E = \emptyset$ and $\mu(E) = \sup\{\sum_{x\in F} f(x) \mid F \text{ is a finite subset of } E\}$ if $E \neq \emptyset$. Show that μ is a measure. *(Marks 4)*.

2. (a) Let $\mathcal{U} = \{A \subset \mathbb{R} \mid A \text{ or } A^c \text{ is countable}\}$ and μ be the measure defined by $\mu(A) = 1$ if A is not countable and $\mu(A) = 0$ if A is countable. Describe the corresponding measurable functions and their integrals *(Marks 4)*.

   (b) Prove that the set of points at which a sequence of measurable functions diverges is measurable.

3. Prove that the Lebesgue outer measure of an interval is its length.

4. (a) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\inf_{E\in\mathcal{A}} \mu(E) > 0$. If f is an integrable function on X, prove that f is bounded a.e.

   (b) State and prove Fatou's lemma *(Marks 3)*.

5. (a) State and prove Borel-Cantelli lemma *(Marks 3)*.

   (b) Let $(X, \mathcal{A}, \mu)$ be a measure space and (A_n) is a descending sequence of sets in $\mathcal{A}$ and $\mu(A_1) < \infty$. Prove that $\mu(A_n) \to \mu(\cap A_n)$.

6. (a) Let $\mathcal{S} = \{\emptyset, X\}$ and define $\mu\colon \mathcal{S} \to [0,\infty)$ by $\mu(\emptyset) = 0, \mu(X) = 1$. Determine μ^*-measurable sets and the Caratheodry measure induced by μ *(Marks 3)*.

   (b) Prove: union of μ^*-measurable sets is μ^*-measurable for a outer measure μ.

7. Let $\overline{\mu}$ be the Caratheodry measure induced by a nonnegative function μ defined on a collection of subsets of a set X and μ^+ be the outer measure induced by $\overline{\mu}$. Show that for $E \subset X$, we have $\mu^+(E) \geq \mu^*(E)$ and $\mu^+(E) = \mu^*(E)$ if and only if there is a μ^*-measurable set A containing E with $\mu^*(A) = \mu^*(E)$.